Analog Computer Applications
Application Note #10
Undergraduate Engineering Physics Space-Themed STEM Project

<div style="text-align:center">

Rocket Propulsion: Volcanic Caldera on Titan
(A Simulation)

</div>

1   Introduction

In this application note, the mission is to determine the speed of a rocket-powered drone sled, as it accelerates across a volcanic caldera icy surface. Quadratic drag is assumed to be the only opposing force while sliding friction is ignored.

Initially, the sled is at rest. After the engine is ignited (powered phase), the sled's speed increases until engine cut off at 20 seconds. Then, the sled simply coasts. However, the coasting phase will not be addressed in this simulation.

For comparison, analytical, numerical, and analog computation solutions are displayed in table 1.

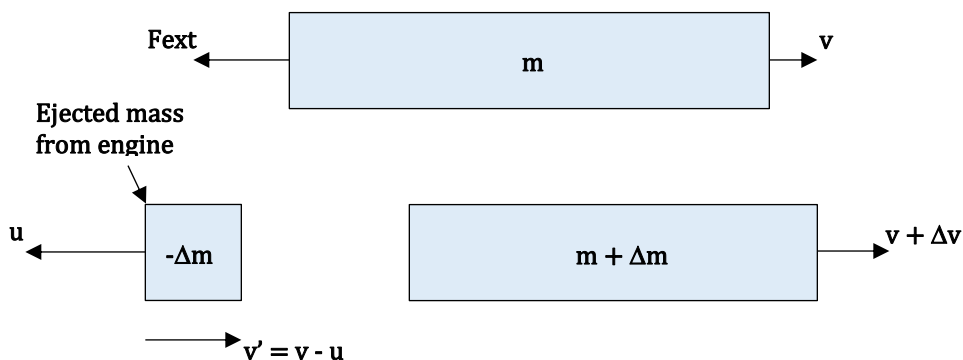# 2   Mathematical modeling

Rocket propulsion equation derivation:

Figure 1: Rocket Propulsion

where,

$$m = \text{mass of rocket and fuel}$$
$$-\Delta m = \text{incremental mass of ejected fuel (note that } \Delta m < 0)$$
$$v = \text{speed of sled relative to a fixed frame of reference}$$
$$\Delta v = \text{incremental speed increase of the rocket and remaining fuel}$$
$$v' = \text{speed of ejected fuel relative to a fixed frame of reference}$$
$$u = \text{speed of ejected fuel relative to the sled (assumed constant)}$$
$$F_{ext} = \text{external forces on rocket (in this case, quadratic drag)}$$

Starting with impulse of the force = change in momentum,

$$F_{ext}\Delta t = \Delta p$$

$$F_{ext}\Delta t = p_{after} - p_{before}$$

$$F_{ext}\Delta t = [(m + \Delta m)(v + \Delta v) + (-\Delta m)(v')] - [mv]$$

$$F_{ext}\Delta t = [(m + \Delta m)(v + \Delta v) + (-\Delta m)(v - u)] - [mv]$$

$$F_{ext}\Delta t = [mv + m\Delta v + v\Delta m + \Delta m\Delta v - v\Delta m + u\Delta m] - [mv]$$

$$F_{ext}\Delta t = m\Delta v + \Delta m\Delta v + u\Delta m$$

$$F_{ext} = m\Delta v/\Delta t + \Delta m\Delta v/\Delta t + u\Delta m/\Delta t$$

$$F_{ext} = m\Delta v/\Delta t + (\Delta m/\Delta t)(\Delta v/\Delta t)\Delta t + u\Delta m/\Delta t$$

$$\lim_{\Delta t\to 0}(F_{ext}) = \lim_{\Delta t\to 0}[m\Delta v/\Delta t + (\Delta m/\Delta t)(\Delta v/\Delta t)\Delta t + u\,\Delta m/\Delta t]$$

$$\lim_{\Delta t\to 0}(F_{ext}) = mdv/dt + udm/dt$$

Assuming $F_{ext} = $ -quadratic drag $ = -0.5C_D\rho Av^2$,

$$\lim_{\Delta t\to 0}(-0.5C_D\rho Av^2) = mdv/dt + udm/dt$$

$$-0.5C_D\rho Av^2 = mdv/dt + udm/dt$$

After a little algebra,

$$dv/dt = (-0.5C_D\rho Av^2 - u\,dm/dt)/m$$

Letting $dm/dt = -c$ (constant), where $= c > 0$ and $m(0) = m_0$,

$$m = -ct + m_0$$

Assuming u is constant,

$$dv/dt = (-0.5C_D\rho Av^2 + uc)/(-ct + m_0)$$

If $m_0 = 1.800$ kg, $c = 0.075$, kg/s, $C_D = 0.250$, $\rho = 4.000$ kg/m³, $A = 0.100$ m², $u = 24$ m/s, and omitting units for simplicity, then

$$dv/dt = (-0.050v^2 + 1.800)/(-0.075t + 1.800) \text{ with } v(0) = 0 \text{ m/s}$$

Use the following equation for the analog computer setup and numerical methods:

$$dv/dt = (-0.050v^2 + 1.800)/(-0.075t + 1.800)$$

For mathematical convenience to avoid decimals, multiply the numerator and denominator of the above equation by -20.

Simplifying,

$$dv/dt = (v^2 - 36)/(3/2\ t - 36)$$

Multiplying the numerator and denominator by 2/3,

$$dv/dt = 2/3(v^2 - 6^2)/(t - 24) \tag{1}$$

Setting $dv/dt = 0$ to determine the sled's terminal speed,

$$0 = 2/3(v_T^2 - 6^2)/(t - 24)$$

By inspection, $v_T = 6$.

Returning to (1),

$dv/dt = 2/3(v^2 - 6^2)/(t - 12)$, which is separable and integrable.

$$dv/(v^2 - 6^2) = 2/3 \, dt/(t - 24)$$

$$\int_0^v dv/(v^2 - 6^2) = 2/3 \int_0^t \frac{dt}{(t - 24)} \qquad (2)$$

Using the method of partial fractions for the integrand on the left,

$$1/(v^2 - 6^2) = A/(v - 6) + B/(v + 6) \qquad (3)$$

$$0V + 1 = (A + B)v + 6(A - B)$$

Comparing coefficients of like terms,

$A + B = 0$ or $B = -A$

and

$6(A - B) = 6(A + A) = 1$ or $A = 1/12$ and $B = -1/12$

Substituting the values for A and B into (3),

$$1/(v^2 - 6^2) = 1/12[1/(v - 6) - 1/(v + 6)] \qquad (4)$$

Substituting (3) into the integrand on the left side of (1),

$$1/12 \int_0^v [1/(v - 6) - 1/(v + 6)]dv = 2/3 \int_0^t dt/(t - 24) \qquad (5)$$

$$\int_0^v [1/(v-6) - 1/(v+6)]dv = 8 \int_0^t dt/(t-24) \qquad (6)$$

Integrating (6) by inspection,

$$\ln|(v-6)/(v+6)|\Big|_0^v = 8 \ln|t-24|\Big|_0^t$$

$$\ln|(v-6)/(v+6)| = 8 \ln|1 - t/24|$$

$$|(v-6)/(v+6)| = e^{\wedge}(8 \ln|1 - t/24|)$$

Since v < 6 and t ≤ 20,

$$-(v-6)/(v+6) = e^{\wedge}(8 \ln(1 - t/24))$$

Letting E = e^(8 ln(1 – t/24)) for simplicity,

$$-(v-6)/(v+6) = E$$

After some algebra,

$$v = 6(1 - E)/(1 + E)$$

Since E > 0, then (1 – E)/(1 + E) < 1. Confirming that v < 6.

Replacing e^(8 ln(1 – t/24) for E,

v= 6(1 − e^(8 ln(1 − t/24)))/(1 + e^(8 ln(1 − t/24)) for t ≤ 20   (7)

3a Computer setup (patch cord version)

-1

1.800

0.075

1

-0.075t + 1.800

$\dfrac{0.050v^2 - 1.800}{-0.075 + 1.800}$

-0.050v² + 1.800

$\dfrac{-0.050v^2 + 1.800}{-0.075t + 1.800}$

v

to oscilloscope

v̇

-v

0

v²   0.050   0.050v²

0.180

-10
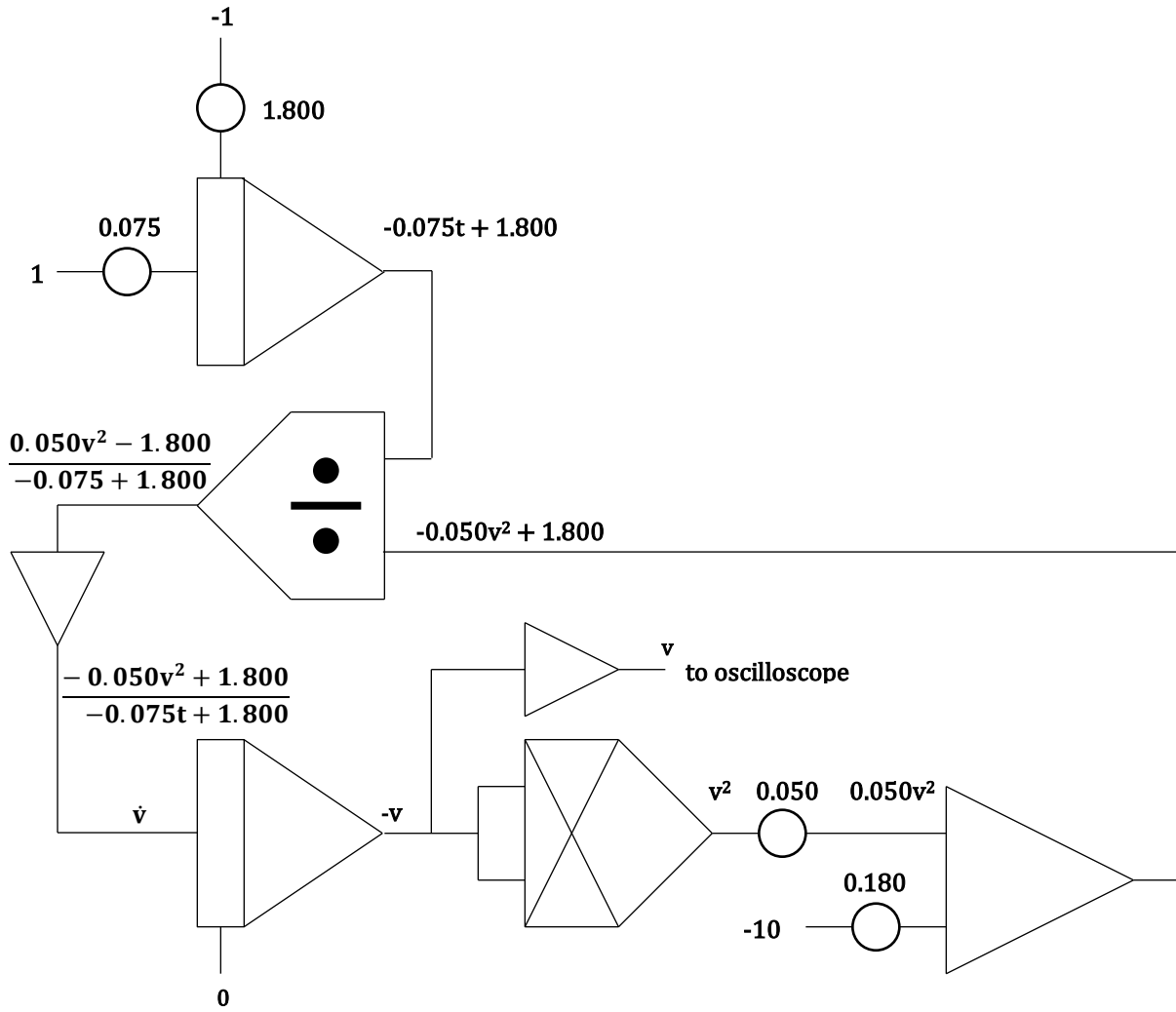
Figure 2: Computer setup for Rocket Propulsion: Volcanic Caldera on Titan

3b Computer setup (op amp/discrete component version)

to linear function generator
voltage divider

reed relay

to linear function generator
output

741

1	8
2	7
3	6
4	5

-dv/dt

reed relay

v

to oscilloscope

1000

1000

(0.075t - 0.900) from divider

Arrow Color Code:

Red arrows connect to V_{CC}.
Blue arrows connect to V_{EE}.
Green arrows connect to ground.

Resistance values in kΩ

Subcircuit Color Code:

Blue
Inverting Integrator

Red
Analog multiplier

Green
Inverting Summer

633

1	8
2	7
3	6
4	5

$v^2/10$

1
μ
F

1
μ
F

220

1	8
2	7
3	6
4	5

741

220

($-0.05v^2$ + 1.800) to divider
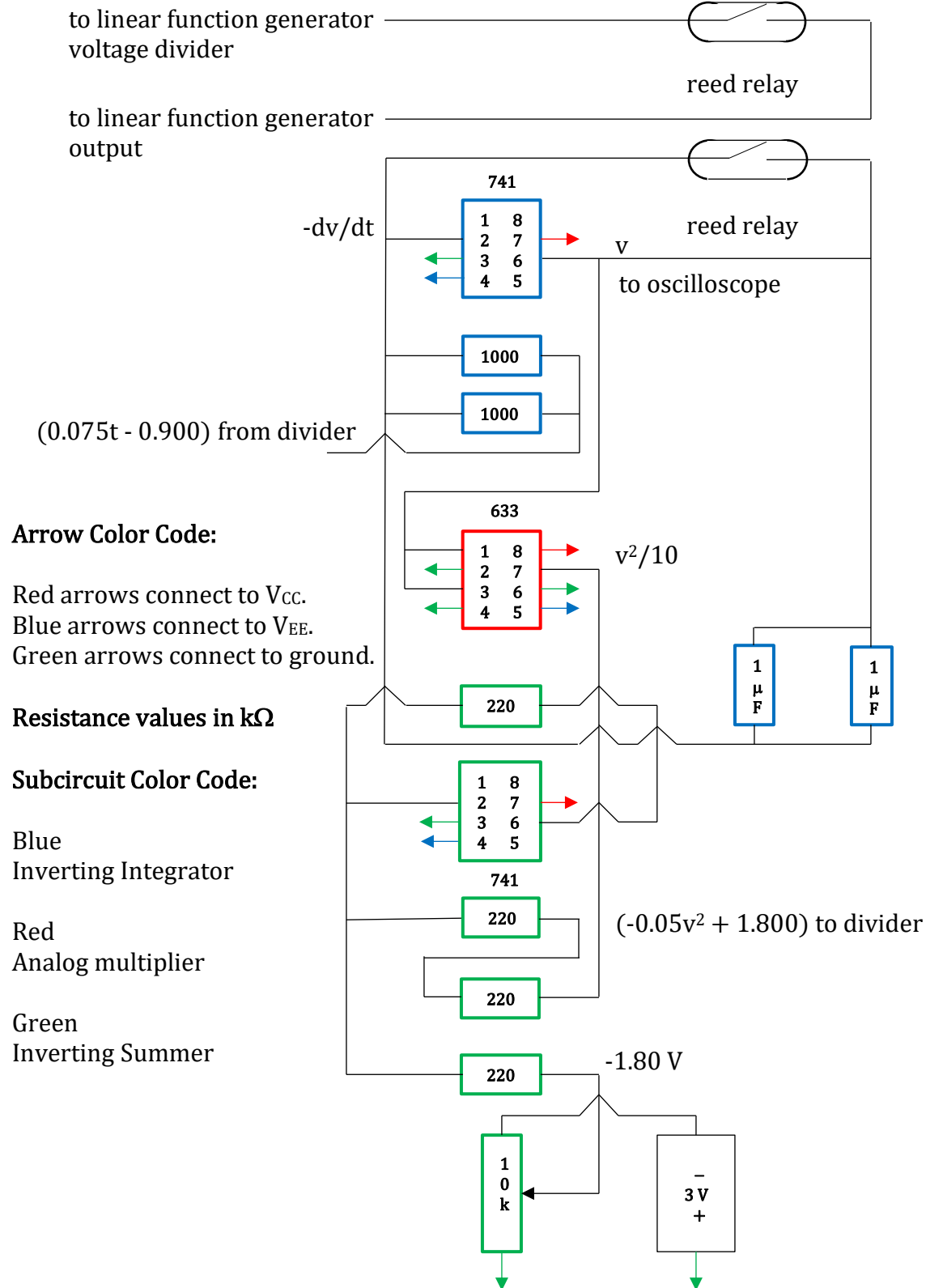
220

220

-1.80 V

1
0
k

–
3 V
+

Figure 3: Basic Breadboard Layout

7

4   Numerical Method

(Modified Euler method using a hand-held programmable calculator)

Code: TI-BASIC

```
PROGRAM:CALDERA
:ClrHome:ClrDraw
:"DV/DT=(-0.050V²+UC)/(-CT+M)"          Note:  M = initial mass
:"INITIAL CONDITION:"
:0→T:0→V
:"ENGINE  PARAMETERS:"
:24→U:0.075→C
:1.800→M:20→E
:"STEP SIZE:"
:0.100→H
:Fix 2
:Lbl 1
:If T>E:Then
:Goto 2:Else
:DisP {T,V}
:(-0.050V²+UC)/(-CT+M)→F
:T+H→T
:V+FH→W
: (-0.050W²+UC)/(-CT+M)→S
:(F+S)/2→A
:V+AH→V
:Pause
:Goto1
:Lbl 2
:End
```

5   Results

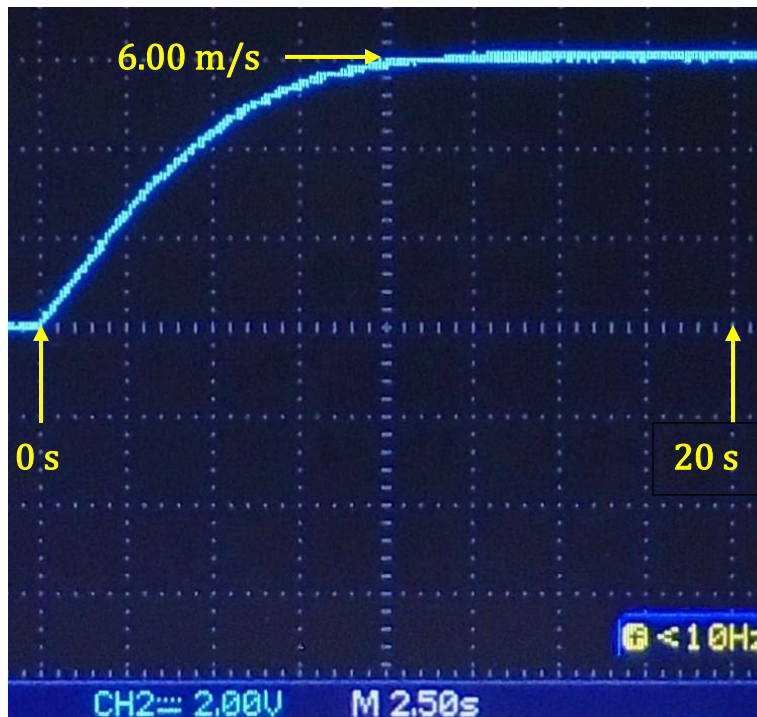| t(s)	Analog computer v(m/s) estimated from oscilloscope	Analytical v(m/s)	Numerical Method v(m/s) Calculator
00.0	0.00	0.00	0.00
02.5	2.48	2.48	2.48
05.0	4.40	4.40	4.40
07.5	5.40	5.43	5.43
10.0	5.90	5.84	5.84
12.5	6.00	5.97	5.97
15.0	6.00	6.00	6.00
17.5	6.00	6.00	6.00
20.0	6.00	6.00	6.00

Table 1: Solution comparisons

Figure 4: Drone sled speed on Titan*

*For this application note, the oscilloscope display was produced during a single run using a differential equation analog computer constructed from operational amplifiers and discrete components with tolerances within 10%.
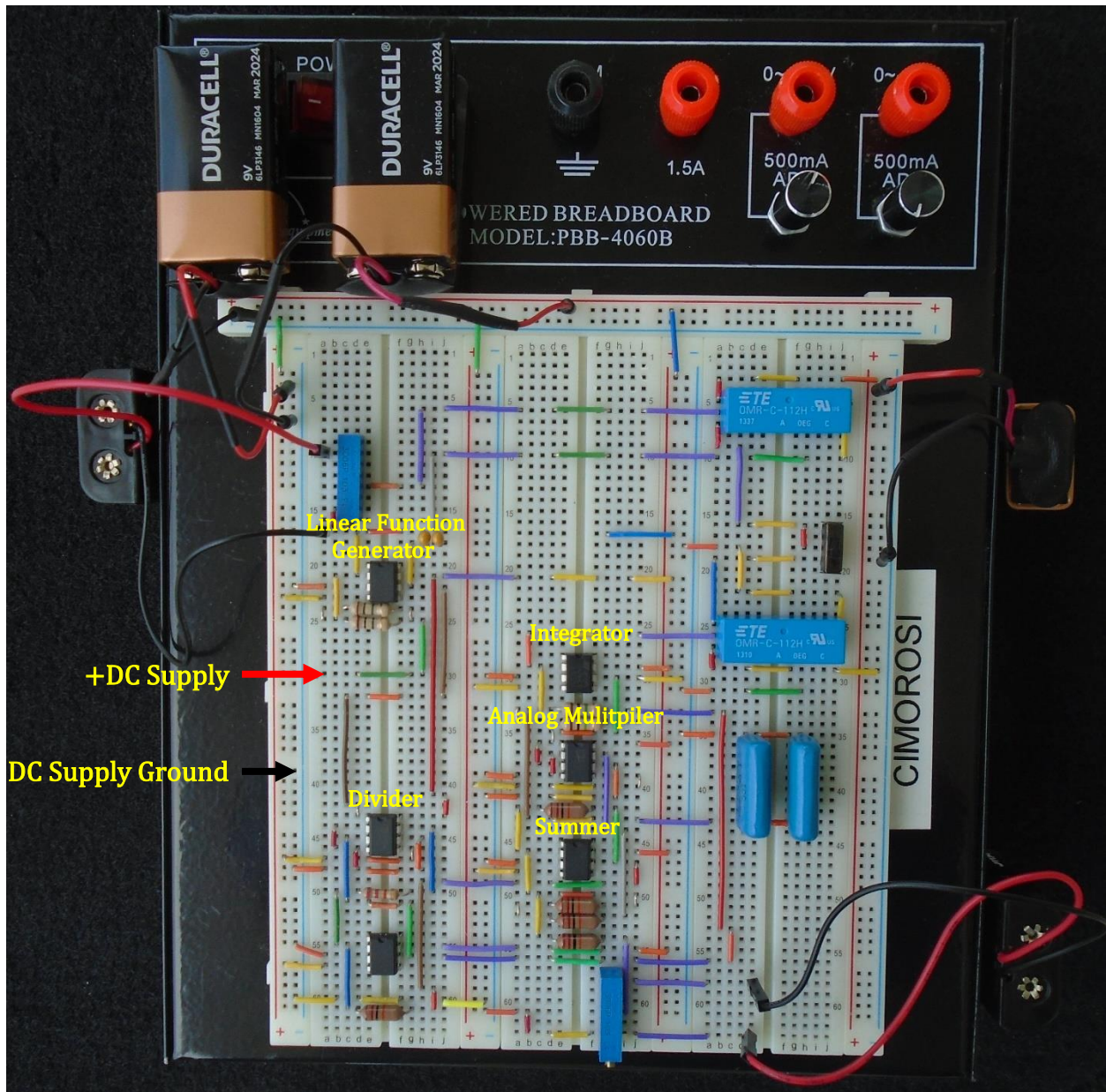
Figure 4: Differential Equation Analog Computer**

** The photograph does not include the DC supply with connections.